Uniqueness of meromorphic functions sharing values with their n[th] order exact differences

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Abstract

Let $f(z)$ be a transcendental meromorphic function in the complex plane with hyper-order strictly less than 1. It is shown that if $f(z)$ and its n[th] exact difference $\Delta_\eta^n f(z)$ $(\not\equiv 0)$ share three distinct periodic functions $a, b, c \in \hat{\mathcal{S}}(f)$ with period η CM, where $\hat{\mathcal{S}}(f) = \mathcal{S}(f) \cup \{\infty\}$ and $\mathcal{S}(f)$ denotes the set of all small functions of $f(z)$, then $\Delta_\eta^n f(z) \equiv f(z)$.

Introduction

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in the complex plane $\mathbb{C}$, and a be a value on $\mathbb{C} \cup \{\infty\}$. We say that f and g share a CM (IM) provided that f and g have the same a-points counting multiplicities (ignoring multiplicities). It is well known that if two nonconstant meromorphic functions f and g share four distinct values CM, then f is a Möbius transformation of g. Rubel and Yang [1] initiated the study of entire functions sharing values with their derivatives, and they proved that $f' \equiv f$ if a nonconstant entire function f and its derivative f' share two distinct finite values CM. This result is then extended for meromorphic function and its arbitrary order derivatives: if a meromorphic function $f(z)$ and its n[th] derivative $f^{(n)}(z)$ share two distinct values $a_1, a_2 \in \mathbb{C}$ CM, then $f^{(n)} \equiv f$.

For the case that $f(z)$ and its n[th] derivative $f^{(n)}$ share two distinct small functions of $f(z)$, Li [2, Theorem 1] obtained that: if a transcendental meromorphic function f shares two distinct small functions CM with its n[th] derivative $f^{(n)}$, $n \geq 2$, then $f^{(n)} \equiv f$. Li [2] also showed that this is not valid generally when $n = 1$.

The main tool in the study of uniqueness of meromorphic functions is Nevanlinna's theory, such as the characteristic function $T(r, f)$, the proximity function $m(r, f)$ and the integrated counting function $N(r, f)$. The notation $S(r, f)$ denotes any quantity that satisfies the condition $S(r, f) = o(1)T(r, f)$ as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure, and a meromorphic function $a(z)$ $(\not\equiv \infty)$ is said to be a *small* function of $f(z)$ if $T(r, a(z)) = S(r, f)$. Denote by $\mathcal{S}(f)$ the field of all small functions of $f(z)$ and set $\hat{\mathcal{S}}(f) = \mathcal{S}(f) \cup \{\infty\}$. For a meromorphic function, the order $\sigma(f)$ the hyper-order $\varsigma(f)$ of $f(z)$ are defined, respectively, by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

and

$$\varsigma(f) = \limsup_{r\to\infty} \frac{\log \log T(r, f)}{\log r}.$$

Letting $\eta \in \mathbb{C} \setminus \{0\}$, we define the shift of $f(z)$ by $f(z+\eta)$ and the exact differences of $f(z)$ by $\Delta_\eta f(z) = f(z+\eta) - f(z)$ and $\Delta_\eta^n f(z) = \Delta_\eta(\Delta_\eta^{n-1} f(z)) = \sum_{i=0}^{n} (-1)^{n-i}\binom{n}{i} f(z + i\eta)$, where n (≥ 2) is an integer. Moreover, we will use the usual notation $\Delta^n f(z)$ when $\eta = 1$.

For finite order meromorphic functions, Halburd and Korhonen [3] and Chiang and Feng [4] proved a difference analogue of the lemma on the logarithmic derivative, independently. Using this new tool, Halburd and Korhonen [5] extended the usual Second Main Theorem for the exact difference $\Delta_\eta f(z)$. By the new type of Second Main Theorem [5, Theorem 2.4], Heittokangas *et. al* [6] proved a shift analogue of the uniqueness theorem on meromorphic function sharing values with its first-order derivative: Let $f(z)$ be a meromorphic function of finite order, and let $\eta \in \mathbb{C}$. If $f(z)$ and $f(z+\eta)$ share three distinct functions $a, b, c \in \hat{\mathcal{S}}(f)$ with period η CM, then $f(z) = f(z+\eta)$ for all $z \in \mathbb{C}$.

Heittokangas *et. al* [7] have improved this result by replacing the condition 3 CM with 2 CM + 1 IM. Moreover, the same conclusion still holds when f has hyper-order $\varsigma(f) < 1$. This can be seen by applying an extension of the difference analogue on the logarithmic derivative lemma for meromorphic functions of hyper-order strictly less than 1 in [8], and by following the same proof.

Since when looking for a difference analogue of the derivative, the exact difference operator is a more natural analogue than the shift operator, a natural question arises: what happens when meromorphic functions of hyper-order strictly less than 1 share values with the n[th] order difference $\Delta_\eta^n f(z)$? Some mathematicians have proved some uniqueness theorems related to this question on finite order meromophic functions sharing values CM with their differences $\Delta_\eta^n f(z)$.

For example, Li [9] proved: if a finite order entire function $f(z)$ and its n[th] difference operator $\Delta_\eta^n f(z)$ share two distinct finite values a_1, a_2 CM and one of the following cases is satisfied: (i) $a_1 a_2 = 0$; (ii) $a_1 a_2 \neq 0$ and $\sigma(f) \notin \mathbb{N}$, then $\Delta_\eta^n f(z) \equiv f(z)$; Li et. al [10] proved: if a finite order entire function $f(z)$ and its difference operator $\Delta_\eta f(z)$ share three distinct values a_1, a_2, a_3 CM in the extended complex plane, then $\Delta_\eta f(z) \equiv f(z)$.

Our paper gives a positive answer to the question posed above by proving the following Theorem 1, which can be viewed as a difference analogue of a result by Li [2, Theorem 1] on the uniqueness of meromorphic functions sharing values with their n[th] order derivatives.

Theorem 1 *Let $f(z)$ be a transcendental meromorphic function of hyper-order strictly less than 1 such that $\Delta^n f(z) \not\equiv 0$. If $f(z)$ and $\Delta^n f(z)$ share three distinct periodic functions $a, b, c \in \hat{\mathcal{S}}(f)$ with period 1 CM, then $\Delta^n f(z) \equiv f(z)$.*

We remark that the linear difference equation

$$\Delta^n f(z) = \sum_{i=0}^{n} (-1)^{n-i}\binom{n}{i} f(z+i) = f(z) \qquad (1)$$

can be solved in terms of exponential functions with coefficients in $\mathcal{P}_1$, which denotes the field of period 1 meromorphic functions defined in $\mathbb{C}$.

Example 2 Since the distinct roots of $\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\lambda^k = 1$ are $\lambda_k = 1 + e^{\frac{2k\pi i}{n}}$, $k = 0, \cdots, n-1$, then the general solution of (1) is $f(z) = \sum_{k=0}^{n-1} \pi_k(z)\lambda_k^z$, $\pi_k(z) \in \mathcal{P}_1$, $k = 0, \ldots, n-1$.

Key step

One of the key lemmas applied in the proof of Theorem 1 is the following lemma from [8], which is a difference analogue of the classical Borel's lemma (see, e.g., [11]) for entire functions with no zeros. We say the zero z_0 of an entire function $g(z)$ with order $i \geq 1$ is *forward invariant* with respect to the translation $\tau(z) = z + \eta$ when z_0 is also a zero of $g(z+\eta)$ with order j and $j \geq i$. For example, all the zeros of an entire function with period η are forward invariant with respect to the translation $\tau(z) = z + \eta$.

Halburd, Korhonen and Tohge [8] proved: let $\eta \in \mathbb{C}$, and $g_0, \ldots, g_n$ be entire functions such that $\varsigma(g_i) < 1$, $i = 0, \ldots, n$ and such that all zeros of $g_0, \ldots, g_n$ are forward invariant with respect to the translation $\tau(z) = z + \eta$. If $g_i/g_j \notin \mathcal{P}_\eta^1$ for all $i, j \in \{0, \ldots, n\}$ such that $i \neq j$, then $g_0, \ldots, g_n$ are linearly independent over $\mathcal{P}_\eta^1$.

We apply this lemma in the case that $abc \not\equiv \infty$, where a suitable transformation gives two functions $g_1(z)$ and $g_2(z)$ of the form

$$g_1(z) = \frac{e^\alpha(1 - e^\beta)}{e^\alpha - e^\beta}, \quad g_2(z) = \frac{1 - e^\beta}{e^\alpha - e^\beta}.$$

and $g_1(z)$, $g_2(z)$ share $0, 1, \infty$. After some calculations we obtain

$$ade^{h_1} + (b - ad)e^{h_2} + \sum_{j=3}^{m} c_j e^{h_j} = 0, \qquad (2)$$

where $c_3, \ldots, c_m$, $(m \in \mathbb{N}^+)$ are nonzero periodic functions of period 1 and

$$h_1 = \sum_{i=0}^{n}(\alpha_i + \beta_i) + \alpha, \quad h_2 = \sum_{i=0}^{n}(\alpha_i + \beta_i) + \beta,$$

and

$$h_j = \sum_{(i_l, i_s) \in S_j} (\alpha_{i_l} + \beta_{i_s}), \quad j \geq 3,$$

and S_j, $j = 3, \ldots, m$ are different subsets of

$$\{(0,1), (0,2), \ldots, (0,n), (1,1), \ldots, (1,n), \ldots, (n,n)\}.$$

Then, by using the above lemma we conclude that e^α and e^β satisfies one of the three relations: (1) $T(r, e^\beta) = S(r, f)$; (2) $T(r, e^\alpha) = S(r, f)$; or (3) $e^\beta = He^{t\alpha}$ for some meromorphic function H such that $T(r, H) = S(r, f)$, but none of these cases can occur implying that $\Delta^n f(z) \equiv f(z)$.

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