Existence of meromorphic solutions of first order difference equations

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Abstract

It is shown that if

$$f(z+1)^n = R(z,f), \qquad (\dagger)$$

where $R(z,f)$ is rational in both arguments, has a transcendental meromorphic solution f of hyper-order < 1, then either f satisfies a difference linear or Riccati equation with rational coefficients, or ($\dagger$) can be transformed into one in a list of five equations which consists of four difference Fermat equations and one equation which is a special case of the symmetric QRT map. In this study, solutions to all of these equations are presented in terms of Weierstrass or Jacobi elliptic functions, or in terms of meromorphic functions which are solutions to a difference Riccati equation. By discarding the assumption that f is of hyper-order < 1, we considered a more general case of admissible meromorphic solutions of ($\dagger$) with meromorphic coefficients and $\deg_f(R(z,f)) = n$.

Introduction

Global existence of large classes of meromorphic solutions is a rare property for a differential equation to have. According to a classical result due to Malmquist, if the first order differential equation

$$f' = R(z,f), \qquad (1)$$

where $R(z,f)$ is rational in both arguments, has a transcendental meromorphic solution, then (1) reduces into the Riccati equation

$$f' = a_2 f^2 + a_1 f + a_0 \qquad (2)$$

with rational coefficients a_0, a_1 and a_2. Generalizations of Malmquist's theorem for the equation

$$(f')^n = R(z,f), \qquad n \in \mathbb{N}, \qquad (3)$$

have been given by Yosida [19] and Laine [8]. Steinmetz [15], and Bank and Kaufman [2] proved that if (3) has rational coefficients and a transcendental meromorphic solution, then by a suitable Möbius transformation, (3) can be either mapped to (2), or to one of the equations in the following list:

$$\begin{aligned}
(f')^2 &= a(f-b)^2(f-\tau_1)(f-\tau_2),\\
(f')^2 &= a(f-\tau_1)(f-\tau_2)(f-\tau_3)(f-\tau_4),\\
(f')^3 &= a(f-\tau_1)^2(f-\tau_2)^2(f-\tau_3)^2,\\
(f')^4 &= a(f-\tau_1)^2(f-\tau_2)^3(f-\tau_3)^3,\\
(f')^6 &= a(f-\tau_1)^3(f-\tau_2)^4(f-\tau_3)^5,
\end{aligned}$$

where a and b are rational functions, and $\tau_1, \ldots, \tau_4$ are distinct constants.

In the second order case, Painlevé [10, 11], Fuchs [3] and Gambier [4] classified all second order differential equations out of the class

$$f'' = F(z,f,f'),$$

where F is rational in f and f' and analytic in z, which have the Painlevé property. Here an ordinary differential equation is said to have the Painlevé property when all solutions are single-valued around all movable singularities. Painlevé and his colleagues ended up with a list of 50 equations, out of which all except 6 could be integrated in terms of known functions, or mapped to another equation within the same list. These six equations are now known as the Painlevé equations, and the prolific study of these equations over the last century has shown their importance both mathematically and in applications.

The existence of globally meromorphic solutions is somewhat more common in the case of difference equations, as compared to differential equations. It was shown by Shimomura [14] that the difference equation $f(z+1) = P(f(z))$, where $P(f(z))$ is a polynomial in $f(z)$ with constant coefficients, always has a non-trivial entire solution. On the other hand, Yanagihara [17] showed that the difference equation

$$f(z+1) = R(f(z)), \qquad (4)$$

where $R(f(z))$ is rational in $f(z)$ having constant coefficients, has a nontrivial meromorphic solution no matter how R is chosen. Moreover, Yanagihara [17] proved that if (4), where $R(z,f(z))$ is now rational in both arguments, has a transcendental meromorphic solution of hyper-order strictly less than one, then $\deg_f(R(z,f(z))) = 1$ and thus (4) reduces into the difference Riccati equation. This is a natural difference analogue of Malmquist's 1913 result on differential equations.

Ablowitz, Halburd and Herbst [1] suggested that the existence of sufficiently many finite-order meromorphic solutions of a difference equation is a good difference analogue of the Painlevé property. They showed, for instance, that if the difference equation

$$f(z+1) + f(z-1) = R(z,f(z)), \qquad (5)$$

where $R(z,f(z))$ is rational in both arguments, has a transcendental meromorphic solution of finite order, then $\deg_f(R(z,f(z))) \le 2$. Halburd and the Korhonen [5] showed that if (5), where the right hand side now has meromorphic coefficients, has an admissible meromorphic solution f of finite order, then either f satisfies a difference Riccati equation, or a linear transformation of (5) reduces it into one in a short list of difference equations which consists solely of difference Painlevé equations and equations related to them, linear equations and linearizable equations. These results appear to verify that the aforementioned approach by Ablowitz, Halburd and Herbst is a good complex analytic difference analogue of the Painlevé property. The finite-order condition was relaxed into hyper-order strictly less than one by Halburd, Korhonen and Tohge [7].

The purpose of our work is to present a natural difference analogue of Steinmetz' generalization of Malmquist's theorem. We prove

Theorem 1 *Let $n \in \mathbb{N}$. If the difference equation*

$$\overline{f}^n = R(z,f), \qquad (6)$$

with rational coefficients has a transcendental meromorphic solution of hyper-order strictly less than 1, then either f satisfies a difference linear or Riccati equation:

$$\begin{aligned}
\overline{f} &= a_1 f + a_2, & (7)\\
\overline{f} &= \frac{b_1 f + b_2}{f + b_3}, & (8)
\end{aligned}$$

where a_i, b_j are rational functions; or, by a transformation $f \to \alpha f$ or $f \to 1/(\alpha f)$ with an algebraic function α of degree at most 3, (6) reduces into one of the following equations:

$$\begin{aligned}
\overline{f}^2 &= 1 - f^2, & (9)\\
\overline{f}^2 &= 1 - \left(\frac{\delta f - 1}{f - \delta}\right)^2, & (10)\\
\overline{f}^2 &= 1 - \left(\frac{f + 3}{f - 1}\right)^2, & (11)\\
\overline{f}^2 &= \frac{f^2 - \kappa^2}{f^2 - 1}, & (12)\\
\overline{f}^3 &= 1 - f^{-3}, & (13)
\end{aligned}$$

where $\delta \not\equiv \pm 1$ is an algebraic function of degree 2 at most and $\kappa^2 \neq 0, 1$ is a constant.

Under the condition that the meromorphic solution f of (6) is of hyper-order less than 1, it can actually be shown that $\deg_f(R(z,f)) = n$ by using an asymptotic relation between the Nevanlinna characteristics $T(r, f(z+1))$ and $T(r, f(z))$ from [7], and an identity due to Valiron [16] (see also [9]). By discarding the assumption that the meromorphic solution is of hyper-order < 1, and considering the more general case of admissible meromorphic solutions of (6) with meromorphic coefficients and $\deg_f(R(z,f)) = n$, it follows either that f satisfies (7) or (8), or (6) can be transformed into one of the equations (9)–(13), but now with meromorphic coefficients $a_i(z)$ and $b_j(z)$, an algebroid function $\delta(z) = \delta_2(z)$ of degree at most 2 and with $\kappa^2 = \kappa_1(z)^2$ being a meromorphic periodic function of period 1, or (6) becomes one of the following equations:

$$\begin{aligned}
f(z+1)^2 &= \delta_1(z)(f(z)^2 - 1), & (14)\\
f(z+1)^2 &= \delta_3(z)(1 - f(z)^{-2}), & (15)\\
f(z+1)^2 &= \frac{\kappa_2(z+1)^2 f(z)^2 - 1}{f(z)^2 - 1}, & (16)\\
f(z+1)^2 &= \theta\frac{f(z)^2 - \kappa_3(z)f(z) + 1}{f(z)^2 + \kappa_3(z)f(z) + 1}, & (17)\\
f(z+1)^3 &= 1 - f(z)^3, & (18)
\end{aligned}$$

where $\theta = \pm 1$ and $\delta_1(z), \delta_3(z), \kappa_2(z)^2, \kappa_3(z)^2$ are meromorpic functions each of which satisfies a certain difference Riccati equation. In particular, if the coefficients of (6) are rational functions, then $\delta_1(z), \delta_3(z), \kappa_2(z)^2, \kappa_3(z)^2$ are all constants. Meromorphic solutions of autonomous versions of (14)–(18) can be characterized by Weierstrass or Jacobi elliptic functions composed with certain entire functions, but none of them is of hyper-order < 1.

Finite order solutions

It is well known that the autonomous versions of the difference linear equation (7) and the difference Riccati equation (8) can be given explicit solutions.

Equations (9)-(11) and (13) are so-called *Fermat* difference equations. For the degree 2 Fermat difference equations (9), (10) and (11), meromorphic solutions to them can be explicitly expressed in terms of functions which are solutions of certain difference Riccati equations. For equation (9), Yanagihara [18] has shown that the solution f can be represented as $f = (\beta + \beta^{-1})/2$, where β satisfies $\overline{\beta} = i\beta^{\pm 1}$. For equation (10), if we put $f = (\gamma + \gamma^{-1})/2$, then γ satisfies

$$\overline{\gamma}^2 - 2i\frac{\delta_2\gamma^2 - 2\gamma + \delta_2}{\gamma^2 - 2\delta_2\gamma + 1}\overline{\gamma} - 1 = 0.$$

and, by solving the above equation, we get

$$\overline{\gamma} = \left\{-\theta\frac{(i\delta_2 - \sqrt{1-\delta_2^2})\gamma + i}{\gamma - \delta_2 + i\sqrt{1-\delta_2^2}}\right\}^{\theta}, \quad \theta = \pm 1.$$

For equation (11), if we put

$$\sqrt{-8}u = \frac{\overline{f}(f-1)}{f+1}, \quad v = \frac{1}{f+1}, \quad \sqrt{2}u = (\lambda + \lambda^{-1})/2,$$

then u satisfies

$$(\sqrt{2}\overline{u})^2 = \frac{2(2u^2-1)}{2u^2 - \sqrt{-8}u - 1} = 1 - \left(\frac{\sqrt{2}iu - 1}{\sqrt{2}u - i}\right)^2,$$

and by combining the solution of (10), we get

$$f = \frac{8\lambda^2 - (\lambda^2+1)^2}{(\lambda^2+1)^2}, \quad \overline{\lambda} = \left\{-\theta\frac{-(1+\sqrt{2})\lambda + i}{\lambda - i + i\sqrt{2}}\right\}^{\theta}, \quad \theta = \pm 1.$$

For the degree 3 Fermat difference equation (13), the meromorphic solution f can be expressed as

$$f = \frac{\wp'(\varphi) - 1/2}{\wp'(\varphi) + 1/2}, \quad \overline{f} = \frac{(-12)^{1/3}\wp(\varphi)}{\wp'(\varphi) - 1/2},$$

where $\varphi = \varphi(z)$ is an entire function such that $\overline{\varphi}' = \eta\varphi'$, $\eta^3 = 1$ and $\wp = \wp(z)$ is the Weierstrass elliptic function satisfying $\wp'(\varphi)^2 = 4\wp(\varphi)^3 - 1/12$. In particular, if $\eta = 1$, then $\overline{\varphi} = \varphi + b$ for a constant b such that $\wp(b) = 12^{-1/3}$ and $\wp'(b) = 1/2$. In this case φ can be a polynomial of degree 1 and thus the order of growth of f is 2.

Equation (12) is a special case of symmetric QRT map [12, 13]. By doing a suitable Möbius transformation, for example, $f \to a(f+1)/(f-1)$, where a is a constant satisfying $2a^4 - 2a^2 - 1 = 0$ and $\kappa_1^2 = a^4$, we get

$$\overline{f}^2 f^2 + \overline{f}^2 + f^2 + 4(1+4a^2)\overline{f}f + 1 = 0,$$

which is solved by a Jacobi elliptic function with finite order of growth [6].

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