Navier-Stokes on manifolds

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ABSTRACT

In recent years there has been a growing interest in studying Navier Stokes equation on manifolds. The main motivation comes from atmospheric models where a two dimensional Navier Stokes equation on the sphere is an important component of the more complete model. There has been two different choices for the diffusion term in the literature. We analyze some consequences of this choice and show that actual solutions behave quite differently depending on this choice.

INTRODUCTION

The standard way to write the Navier Stokes equations in $\mathbb{R}^n$ is as follows

$$u_t + u\nabla u - \nu\Delta u + \nabla p = f$$
$$\nabla \cdot u = 0$$

Let us formulate this on an arbitrary Riemannian manifold M with Riemannian metric g. Let ∇ now denote the covariant derivative, and to avoid confusion we write $\mathsf{grad}(p)$ for the gradient and $\mathsf{div}(u)$ for the divergence.

The diffusion term Δu is a bit problematic since there are various ways to generalize the Laplacian for vectors. It seems that there are two common choices for this when analyzing Navier-Stokes equations. The first is the *Bochner Laplacian*, defined by the formula

$$\Delta_B u = g^{ij} u^k_{;ij}$$

and the second one is

$$Lu = \mathsf{div}(Su) = \mathsf{div}\big(g^{ki}u^j_{;i} + g^{ij}u^k_{;i}\big)$$

Su can be thought of as twice the deformation rate tensor. This gives the Navier Stokes system

$$u_t + \nabla_u u - \nu\,\Delta u + \mathsf{grad}(p) = f$$
$$\mathsf{div}(u) = 0$$

where Δ is either L or Δ_B. We will analyze some consequences of this choice.

The difference of these operators can be characterized as follows.

Lemma 1

$$Lu = \Delta_B u + \mathsf{grad}\big(\mathsf{div}(u)\big) + \mathsf{Ri}(u)$$

where Ri *is the* Ricci tensor.

In $\mathbb{R}^n$ the Ricci tensor is zero so that $Lu = \Delta_B u$ for divergence free vector fields. However, for manifolds where the curvature tensor is nontrivial, in particular for the sphere, these operators are not equivalent.

KILLING VECTOR FIELDS

v is a *Killing vector field* if $Sv = 0$. Note that this implies that $\mathsf{div}(v) = 0$. Moreover one can show that $\mathsf{div}\big(\mathsf{Ri}(v)\big) = 0$ if v is Killing. Killing vector fields exist in many interesting situations.

Theorem 2 *Let M be a compact n dimensional Riemannian manifold without boundary. Then the Killing vector fields are a Lie algebra whose dimension is $\leq \frac{1}{2}n(n+1)$ and the equality is attained for the standard sphere.*

The Killing fields are important in the analysis of Navier Stokes equations on the manifold because they are stationary solutions for the Navier Stokes equations.

Theorem 3 *Let M be a compact n dimensional Riemannian manifold without boundary. The homogeneous Navier Stokes system with the operator L has solutions of the form (v,p) where v is Killing and $p = \frac{1}{2}g(v,v) + c$ where c is constant.*

Killing vector fields on the sphere correspond to the rotating flows. Note that these are not solutions with the Bochner Laplacian.

In numerical computations one has to solve the linearized version of Navier Stokes:

$$u_t + \nabla_v u - \nu\,Lu + \mathsf{grad}(p) = 0$$
$$\mathsf{div}(u) = 0$$

where v is some given vector field. The Killing vector fields give us new solutions from the given ones with help of the Lie bracket.

Theorem 4 *Suppose that u is a solution to the linearized Navier Stokes system and that v is a Killing vector field. Then $w = [u,v]$ is also a solution.*

To prove this one needs to establish several results. First the new pressure is given by

$$\hat{p} = -g\big(\mathsf{grad}(p), v\big)$$

from which it follows that $\mathsf{grad}(\hat{p}) = [\mathsf{grad}(p), v]$. Then we need the formula

$$\nabla_v[u,v] = [\nabla_v u, v] \tag{1}$$

Here we need the fundamental identity which characterizes the second derivatives of the Killing vector field in terms of the curvature tensor:

$$\nabla^2_{X,Y} v = R(Y,v)X$$
$$v^i_{;hk} = -v^\ell R^i_{\ell kh}$$

Using this we can prove (1) by computing

$$\begin{aligned}\nabla_v[u,v] &= \nabla_v\nabla_u v - \nabla_v\nabla_v u\\ &= \nabla_{(\nabla_v u)}v + v^k u^h v^i_{;hk} - \nabla_v(\nabla_v u)\\ &= [\nabla_v u, v] + v^k u^h v^i_{;hk}\end{aligned}$$

However, since v is Killing

$$v^k u^h v^i_{;hk} = R(v,v)u = 0$$

Similar, but more difficult computations show that

$$\begin{aligned}\Delta_B[u,v] &= [\Delta_B u, v]\\ \mathsf{Ri}[u,v] &= [\mathsf{Ri}(u), v]\end{aligned}$$

ENERGY OF THE SOLUTION

S induces a linear map $S_u : T_pM \to T_pM$ given by

$$S_u v = \big(g^{ki}u^j_{;i} + g^{ij}u^k_{;i}\big)g_{j\ell}v^\ell = \big(g^{ki}u^j_{;i}g_{j\ell} + u^k_{;\ell}\big)v^\ell$$

Note that S_u is symmetric: $g(S_u v, w) = g(v, S_u w)$. The metric induces an inner product also for general tensors so that one can write $g(\nabla u, \nabla v)$.

Let us set

$$\langle u, v\rangle = \int_M g(u,v)\omega_M$$

where ω_M is the volume form. This defines the usual L_2 norm for vector fields.

Theorem 5 *The energy of the flow is governed by the following formulas*

$$\tfrac{1}{2}\partial_t\|u\|^2 + \int_M g(\nabla u, \nabla u)\omega_M = 0$$
$$\partial_t\|u\|^2 + \int_M g(S_u, S_u)\omega_M = 0$$

where in the first line Bochner Laplacian is used and in the second line the operator L is used.

Let us outline the proof in case of L. We compute

$$\begin{aligned}\mathsf{div}(S_u v) &= g^{ki}u^j_{;ik}g_{j\ell}v^\ell + g^{ki}u^j_{;i}g_{j\ell}v^\ell_{;k} + u^k_{;\ell k}v^\ell + u^k_{;\ell}v^\ell_{;k}\\ &= \tfrac{1}{2}g(S_u, S_v) + g(Lu, v)\end{aligned}$$

Hence by the divergence theorem

$$\int_M g(Lu,v)\omega_M + \tfrac{1}{2}\int_M b_S(u,v)\omega_M = 0$$

Moreover if $\mathsf{div}(u) = 0$ then

$$\int_M g(\nabla_u u, u)\omega_M = 0$$
$$\int_M g\big(\mathsf{grad}(p), u\big)\omega_M = 0$$

From this result it follows that the solutions behave very differently depending on the choice of the diffusion term.

VORTICITY

Vorticity of the flow is important in the analysis of the properties of the solution. Let us only consider the two dimensional case. First we define the tensor

$$\varepsilon = \sqrt{\det(g)}\big(dx_1 \otimes dx_2 - dx_2 \otimes dx_1\big)$$

and the operator $Ku = g^{ik}\varepsilon_{k\ell}u^\ell = \varepsilon^i_\ell u^\ell$. Then the vorticity ζ is given by

$$\zeta = \mathsf{div}(Ku) = \varepsilon^i_\ell u^\ell_{;i}$$

Then we can show

Theorem 6 *If u is the solution of the Navier-Stokes equation, then*

$$\zeta_t - \nu\,\Delta\zeta + g\big(\mathsf{grad}(\zeta), u\big) - c\,\nu\,g\big(\mathsf{grad}(\kappa), Ku\big) - c\,\nu\,\kappa\zeta = 0$$

where κ is the Gaussian curvature and $c = 2$ if the operator L is used for the diffusion and $c = 1$ for the Bochner Laplacian.

This again shows that the choice of the diffusion term has a strong effect on the properties of the solution.

References

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