Linear differential equations with solutions in the growth space H_ω^∞

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Abstract

We give sufficient conditions for analytic coefficients A_k of

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = A_n(z)$$

such that all solutions or their derivatives belong to H_ω^∞. Here H_ω^∞ consists of those analytic functions f in the unit disc $\mathbb{D}$ for which $|f(z)|\omega(z)$ is uniformly bounded, and $\omega : \mathbb{D} \to (0,\infty)$ is radial and measurable and satisfies certain regularity conditions.

Introduction

We study the growth of solutions of the differential equation

$$f^{(n)}+A_{n-1}(z)f^{(n-1)}+\cdots+A_1(z)f'+A_0(z)f = A_n(z), \quad n \geq 2, \tag{1}$$

where $A_0(z), A_1(z), \ldots, A_n(z)$ are analytic in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$, denoted by $A_0, A_1, \ldots, A_n \in \mathcal{H}(\mathbb{D})$ for short. In particular, we are interested in the case where the solutions or their derivatives belong to

$$H_\omega^\infty = \left\{ g \in \mathcal{H}(\mathbb{D}) : \|g\|_{H_\omega^\infty} := \sup_{z\in\mathbb{D}} |g(z)|\omega(z) < \infty \right\}.$$

Here ω is a (radial) weight, which means that $\omega : \mathbb{D} \to (0,\infty)$ is bounded, measurable and satisfies $\omega(z) = \omega(|z|)$ for all z. The case $\omega(z) = (1-|z|)^p$, $p \in (0,\infty)$ is denoted simply by H_p^∞. We also consider the derivatives of the solutions and denote $\mathcal{B}^\alpha = \{f : f' \in H_\alpha^\infty\}$ with $\mathcal{B} = \mathcal{B}^1$ being the classical Bloch space.

The growth of solutions of (1) depends almost entirely on the growth of the coefficients A_k. Consider, for example, the differential equation

$$f'' + A(z)f = 0. \tag{2}$$

Then all solutions are bounded when $A \in H_{2-\varepsilon}^\infty$ for any $\varepsilon \in (0,2)$ [6, Corollary 3.16]. On the other hand, if $A \in H_{2+\varepsilon}^\infty \setminus \cup_{p<2+\varepsilon} H_p^\infty$ for $\varepsilon \in (0,\infty)$, then the order of growth of any nontrivial solution is $\varepsilon/2$ [2, Theorem 1.4(c)]. To ensure that the growth of the solutions is somewhere between these two extremal cases, the growth condition for the coefficient $A(z)$ needs to be more delicate. For example, if $\|A\|_{H_2^\infty}$ or $\sup_{z\in\mathbb{D}}[|A(z)|(1-|z|)^2|\log(1-|z|)|]$ is small enough, then all solutions belong to H_p^∞ or $\mathcal{B}$ (respectively), see Example 5.

Conditions on weights

We consider weights ω and ω_k that satisfy the conditions

$$\limsup_{r\to1^-} \frac{\omega(r)}{\omega\left(\frac{1+\varepsilon r}{1+\varepsilon}\right)} < m \tag{3}$$

for some constants $\varepsilon \in (0,\infty)$ and $m = m(\omega,\varepsilon) \in (0,\infty)$, and

$$\limsup_{r\to1^-} \int_0^r \frac{ds}{\omega_k(s)}\,\omega_{k-1}(r) < M_k < \infty, \quad k = 1,2,\ldots,n, \tag{4}$$

where $M_k = M_k(\omega_k, \omega_{k-1}) > 0$. Regarding the constants M_k we also write $P_n := \prod_{k=1}^n M_k$. It should be noted that conditions (3) and (4) are independent and have the following properties.

(i) It is possible that (3) holds for some ε but not for all. However, if (3) holds for some ε, then it holds for some arbitrarily small ε.

(ii) If ω is nonincreasing and (3) holds for some ε, then it holds for all ε. Hence, in this case (3) is equivalent to the doubling condition $\omega(r) \leq m\omega\left(\frac{1+r}{2}\right)$ when $r \in [0,1)$ is large enough.

(iii) If $\mu : [0,1) \to (0,\infty)$ is nonincreasing and ω satisfies (4), then $\omega_k\mu$ satisfies (4) with $M' \leq M_k$. Moreover, if $\omega\mu^k$ satisfies (4) for $k = 1$, then it satisfies the condition for all $k \in \mathbb{N}$ with a nonincreasing sequence of constants $(M_k)_{k=1}^\infty$. The typical example $\omega_k(r) = \omega(r)(1-r)^k$ is obtained by the choice $\mu(r) = 1-r$.

General case

In what follows, we will use the notation

$$\|g\|_{H_{\omega,\mu}^\infty} = \sup_{z\in\mathbb{D}} |g(z)|\omega(z) \int_0^{|z|} \frac{dr}{\mu(r)},$$

where $g \in \mathcal{H}(\mathbb{D})$ and ω and μ are weights.

The following theorem is a simplified version of our main result.

Theorem 1. *Let f be a solution of (1) where $A_n \equiv 0$, and suppose that ω is a weight satisfying (3). Denote $\omega_0 = \omega$ and $\omega_k(z) = \omega(z)(1-|z|)^k$ for $k \in \mathbb{N}$. Then the following assertions hold:*

(a) *If ω_k satisfies (4) for all $k = 1,2,\ldots,n$, and*

$$E := P_n\left(\|A_0\|_{H_\omega^\infty} + m\sum_{k=1}^{n-1} k!(1+\varepsilon)^k \|A_k\|_{H_{\omega_{n-k}}^\infty} \right) < 1,$$

where m is as in (3), then $f \in H_\omega^\infty$.

(b) *If ω_k satisfies (4) for all $k = 1,2,\ldots,n-1$, and*

$$F := P_{n-1}\Bigg(\|A_0\|_{H_{\omega_{n-1},\omega}^\infty} + \|A_1\|_{H_{n-1}^\infty} + m\sum_{k=1}^{n-2} k!(1+\varepsilon)^k \|A_{k+1}\|_{H_{n-k-1}^\infty} \Bigg) < 1,$$

where m is as in (3), then $f' \in H_\omega^\infty$.

We make the following remarks about Theorem 1.

(i) Theorem 1 generalizes to the non-homogenous equation (1):
  1. In Theorem 1(a) the condition $A_n \equiv 0$ can be replaced by the condition $A_n \in H_{\omega_n}^\infty$.
  2. In Theorem 1(b) the condition $A_n \equiv 0$ can be replaced by the condition $A_n \in H_{\omega_{n-1}}^\infty$.

(ii) If one of the following conditions holds, then the assumption that ω satisfies (3) is not necessary.
  1. In Theorem 1(a) $A_{n-1} \equiv A_{n-2} \equiv \ldots \equiv A_1 \equiv 0$.
  2. In Theorem 1(b) $A_{n-1} \equiv A_{n-2} \equiv \ldots \equiv A_2 \equiv 0$.

Special cases and examples

We denote by $H_{p,q}^\infty$ the space of functions $g \in \mathcal{H}(\mathbb{D})$ such that

$$\|g\|_{H_{p,q}^\infty} = \sup_{z\in\mathbb{D}} \left[|g(z)|(1-|z|)^p I_q(z) \right] < \infty,$$

where $p, q \in (0,\infty)$ and

$$I_q(z) = \int_0^{|z|} \frac{ds}{(1-s)^q} = \begin{cases} \frac{1}{1-q}\left[1-(1-|z|)^{1-q}\right], & q \in (-\infty,1), \\ \log\frac{1}{1-|z|}, & q = 1, \\ \frac{1}{q-1}\left[\frac{1}{(1-|z|)^{q-1}} - 1\right], & q \in (1,\infty). \end{cases}$$

Note that $H_{p,q}^\infty = H_{p+1-q}^\infty$ when $1 < q < p+1$, and $H_{p,q}^\infty = H_p^\infty$ when $q \in (0,1)$ and $p \in (0,\infty)$.

The following result is an important special case of Theorem 1.

Theorem 2. *Let f be a solution of the differential equation (1) where $A_n \equiv 0$. Then the following assertions hold:*

(a) *If, for $p \in (0,\infty)$,*

$$E := \prod_{j=1}^n \frac{1}{p+j-1}\left(\|A_0\|_{H_n^\infty} + \sum_{k=1}^{n-1} k!\frac{(k+p)^{k+p}}{k^k p^p} \|A_k\|_{H_{n-k}^\infty} \right) < 1,$$

then

$$\|f\|_{H_p^\infty} \leq \frac{|f(0)| + \sum_{k=1}^{n-1}\prod_{j=1}^{k-1}\frac{1}{p+j-1}|f^{(k)}(0)|}{1-E}.$$

(b) *If, for $\alpha \in (0,\infty)$,*

$$F := \prod_{j=1}^{n-1} \frac{1}{\alpha+j-1}\Bigg(\|A_0\|_{H_{\alpha+n-1,\alpha}^\infty} + \|A_1\|_{H_{n-1}^\infty} + \sum_{k=1}^{n-2} k!\frac{(k+\alpha)^{k+\alpha}}{k^k\alpha^\alpha} \|A_{k+1}\|_{H_{n-k-1}^\infty} \Bigg) < 1,$$

then

$$\|f\|_{\mathcal{B}^\alpha} \leq \frac{\prod_{j=1}^{n-1}\frac{1}{\alpha+j-1}\|A_0\|_{H_{\alpha+n-1}^\infty}|f(0)| + |f'(0)|}{1-F} + \frac{\sum_{k=2}^{n-1}\prod_{j=1}^{k-1}\frac{1}{\alpha+j-1}|f^{(k)}(0)|}{1-F}.$$

By Theorem 2 we easily obtain the following result regarding the important special case (2) of (1), where the coefficient A is given by a power series.

Corollary 3. *Let f be a solution of the differential equation (2), where $A(z) = \sum_{k=0}^\infty a_k z^k \in \mathcal{H}(\mathbb{D})$. Then the following assertions hold:*

(a) *If $\alpha \in (0,1)$ and $|a_k| < \alpha(1-\alpha)\dfrac{\Gamma(k+\alpha+1)}{k!\,\Gamma(\alpha+1)}$ for $k \in \mathbb{N}\cup\{0\}$, then $f \in \mathcal{B}^\alpha$.*

(b) *If $|a_k| < \dfrac{1}{k!}\displaystyle\int_1^2 \frac{\Gamma(k+x)}{\Gamma(x)}dx$ for $k \in \mathbb{N}\cup\{0\}$, then $f \in \mathcal{B}$.*

(c) *If $\alpha \in (1,\infty)$ and $|a_k| < \alpha(\alpha-1)(1+k)$ for $k \in \mathbb{N}\cup\{0\}$, then $f \in \mathcal{B}^\alpha$.*

Using [7, Theorem 14] we can also state the following consequence of Theorem 2 which concerns the case where $A(z)$ is a gap series.

Corollary 4. *Let f be a solution of the differential equation (2), where*

$$A(z) = C\sum_{k=0}^\infty a_k z^{n_k}, \quad 1 < q \leq \frac{n_{k+1}}{n_k}, \quad k, n_k \in \mathbb{N},$$

and $C > 0$ is a constant independent of z. Then the following assertions hold:

(a) *If C is small enough and $\limsup_{k\to\infty}|a_k|n_k^{-1-\alpha} < \infty$ for $\alpha \in (0,1)$, then $f \in \mathcal{B}^\alpha$.*

(b) *If C is small enough and $\limsup_{k\to\infty}|a_k|n_k^{-2}\log n_k < \infty$, then $f \in \mathcal{B}$.*

(c) *If C is small enough and $\limsup_{k\to\infty}|a_k|n_k^{-2} < \infty$, then $f \in \mathcal{B}^\alpha$ for $\alpha \in (1,\infty)$.*

We conclude with an example showing that Theorems 1 and 2 are sharp in the sense that the assumptions $E < 1$ and $F < 1$ cannot be relaxed to $E < 1+\varepsilon$ and $F < 1+\varepsilon$, respectively, for any $\varepsilon \in (0,\infty)$.

Example 5. *Let us consider the differential equation (2).*

(a) *If $A(z) = -(p+\alpha)(p+\alpha+1)(1-z)^{-2}$ for $p \in (0,\infty)$ and $\alpha \in [0,\infty)$, then (2) has a solution base $\{f_1, f_2\}$, where*

$$f_1(z) = (1-z)^{-p-\alpha} \quad \text{and} \quad f_2(z) = (1-z)^{p+\alpha+1}.$$

Hence, if $\alpha = 0$, then $\|A\|_{H_2^\infty}/p(p+1) = 1$ and all solutions belong to H_p^∞ space. On the other hand, for any $\varepsilon \in (0,\infty)$, we find $\alpha = \alpha(\varepsilon) \in (0,\infty)$ such that $\|A\|_{H_2^\infty}/p(p+1) \in (1, 1+\varepsilon)$ and $f_1 \notin H_p^\infty$.

(b) *If $A(z) = -\alpha(1-z)^{-2}\left((\alpha-1)\left(\log\frac{e}{1-z}\right)^{-2} + \left(\log\frac{e}{1-z}\right)^{-1}\right)$ for $\alpha \in [1,\infty)$, then (2) has a solution base $\{f_1, f_2\}$, where*

$$f_1(z) = \left(\log\frac{e}{1-z}\right)^\alpha \quad \text{and} \quad f_2(z) = f_1(z)\int_0^z \frac{d\zeta}{f_1(\zeta)^2}.$$

Here $|f_2'(z)| \leq \left(\log\frac{e}{2}\right)^{-\alpha}|f_1'(z)| + \left(\log\frac{e}{2}\right)^{-2\alpha}$ for $z \in \mathbb{D}$. Hence, for $\alpha = 1$, we have $\|A\|_{H_{2,1}^\infty} = 1$ and that all solutions belong to $\mathcal{B}$. However, for any $\varepsilon \in (0,\infty)$, we find $\alpha = \alpha(\varepsilon) \in (1,\infty)$ such that $\|A\|_{H_{2,1}^\infty} \in (1, 1+\varepsilon)$ and $f_1 \notin \mathcal{B}$.

References

[1] I. Chyzhykov, G. Gundersen and J. Heittokangas, Linear differential equations and logarithmic derivative estimates, Proc. London Math. Soc. 86 (2003), 735–754.

[2] I. Chyzhykov, J. Heittokangas and J. Rättyä, Sharp logarithmic derivative estimates with applications to ordinary differential equations in the unit disc. J. Aust. Math. Soc. 88 (2010), 145–167.

[3] J. Heittokangas, On complex differential equations in the unit disc, Ann. Acad. Sci. Fenn. Math. 122 (2000), 1–54.

[4] J. Heittokangas, R. Korhonen and J. Rättyä, Fast growing solutions of linear differential equations in the unit disc, Results Math. 49(34) (2006), 265–278.

[5] J. Heittokangas, R. Korhonen and J. Rättyä, Growth estimates for solutions of nonhomogeneous linear complex differential equations, Ann. Acad. Sci. Fenn. Math. 34 (2009), 145156.

[6] J. Heittokangas, R. Korhonen and J. Rättyä, Linear differential equations with solutions in the Dirichlet type subspace of the Hardy space, Nagoya Math. J. 187 (2007), 91–113.

[7] J. Peláez and J. Rättyä, Generalized Hilbert operators on weighted Bergman spaces. Adv. Math. 240 (2013), 227267.

[8] J. Peláez and J. Rättyä, Weighted Bergman spaces induced by rapidly increasing weights. Mem. Amer. Math. Soc. 227 (2014).

[9] Ch. Pommerenke, On the mean growth of the solutions of complex linear differential equations in the disk, Complex Var. Theory Appl. 1 (1982), 23–38.