Uniformly separated zeros of solutions of ODEs

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ABSTRACT

Sufficient conditions on the analytic coefficient A are considered, which imply uniform separation for the zeros of all non-trivial solutions of $f''+Af=0$. The conditions are given in terms of Carleson measures.

INTRODUCTION

The sequence $\{z_n\}$ of points in the unit disc $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ is called *uniformly separated* if

$$\inf_k \prod_{n\neq k}\left|\frac{z_n-z_k}{1-\overline{z}_n z_k}\right|>0,$$

while $\{z_n\}\subset\mathbb{D}$ is said to be *separated*, with respect to the hyperbolic metric, if there exists a constant $\delta=\delta(\{z_n\})>0$ such that

$$\varrho(z_n,z_k)=\left|\frac{z_n-z_k}{1-\overline{z}_n z_k}\right|>\delta,\quad n\neq k. \tag{1}$$

The set

$$Q=Q(I)=\{re^{i\theta}:e^{i\theta}\in I,\,1-|I|\leq r<1\}$$

is called a *Carleson square* based on the arc $I\subset\partial\mathbb{D}$, where $|I|=\ell(Q)$ denotes the normalized arc length of I. The top part of $Q(I)$ is

$$T\big(Q(I)\big)=\{re^{i\theta}:e^{i\theta}\in I,\,1-|I|\leq r<1-|I|/2\}.$$

For $0<K<\infty$, KQ denotes the Carleson square whose base is concentric to that of Q, and for which $\ell(KQ)=K\,\ell(Q)$.

A finite positive measure μ in $\mathbb{D}$ is said to be a *Carleson measure*, if there exists a constant $0<M<\infty$ such that

$$\int_{\mathbb{D}}|f(z)|\,d\mu(z)\leq M\left(\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|\,d\theta\right)$$

for any analytic function f in the unit disc. Carleson proved that this holds if and only if there exists a constant $0<C<\infty$ such that

$$\mu(Q)\leq C\,\ell(Q)$$

for any Carleson square Q. Moreover, it is known that μ is a Carleson measure if and only if μ is a positive measure for which

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{1-|a|^2}{|1-\overline{a}z|^2}\,d\mu(z)<\infty.$$

The sequence $\{z_n\}\subset\mathbb{D}$ is uniformly separated if and only if it is separated and there exists a constant $0<C<\infty$ such that

$$\sum_{z_n\in Q}(1-|z_n|)\leq C\,\ell(Q) \tag{2}$$

for any Carleson square Q. For more details on Carleson measures, see [2].

RESULTS

By the classical results [7] due to B. Schwarz it is known that

$$\|A\|_{H_2^\infty}=\sup_{z\in\mathbb{D}}(1-|z|^2)^2\,|A(z)|<\infty$$

if and only if the zero-sequence of each non-trivial solution f of

$$f''+Af=0 \tag{3}$$

is separated (by a constant depending only on $\|A\|_{H_2^\infty}$). For the interplay between the maximal growth of the coefficient A and the minimal separation of the zeros of non-trivial solutions of (3), we refer to [1].

Theorem 1 (See [4]). *If the coefficient A is analytic in $\mathbb{D}$ and $|A(z)|(1-|z|^2)\,dm(z)$ is a Carleson measure, then the zero-sequence of each non-trivial solution of (3) is uniformly separated.*

The proof of Theorem 1 relies on Jensen's formula and a certain growth estimate [5] according to which all solutions f of (3) belong to the Nevanlinna class N if A is analytic in $\mathbb{D}$ and

$$\int_{\mathbb{D}}|A(z)|(1-|z|^2)\,dm(z)<\infty.$$

The Nevanlinna class N consists of those analytic functions in $\mathbb{D}$ for which

$$\sup_{0\leq r<1}m(r,f)=\sup_{0\leq r<1}\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta<\infty.$$

Uniform separation through intermediate points

If $z,w\in\mathbb{D}$ are two distinct points, then we define $\langle z,w\rangle\subset\mathbb{D}$ to be the hyperbolic segment joining z and w. That is, $\langle z,w\rangle$ is a closed subarc of the unique hyperbolic geodesic which goes through $z\in\mathbb{D}$ and $w\in\mathbb{D}$.

The following result shows that a separated sequence of points is in fact uniformly separated if there exists a sufficiently dispersed intermediate sequence.

Theorem 2 (See [3]). *Let $\{z_n\}$ be a separated sequence of points in $\mathbb{D}$. Suppose that there exists a sequence $\Lambda\subset\mathbb{D}$ satisfying the following properties:*

- (i) *for each pair of distinct points $z_j,z_k\in\{z_n\}$ there corresponds a point $\xi_{j,k}\in\Lambda$ such that $\xi_{j,k}\in\langle z_j,z_k\rangle$;*
- (ii) *each separated subsequence of Λ is uniformly separated.*

Then, $\{z_n\}$ is uniformly separated.

Sketch of the proof of Theorem 2. By breaking $\{z_n\}$ into finitely many subsequences, we may assume the following two conditions:

- (A) $\{z_n\}$ satisfies (1) for $0<\delta<1$, where δ is so large that the top part of each Carleson square contains at most one point from $\{z_n\}$;
- (B) $\{z_n\}$ satisfies

$$\{z_n\}\subset\bigcup_{k=1}^\infty\{z\in\mathbb{D}:2^{-7k}<1-|z|\leq 2^{-(7k-1)}\}.$$

We show that there exists a constant $0<C<\infty$ such that (2) holds for any Carleson square Q for which $0<\ell(Q)<1/8$. Let Q be a such Carleson square. By means of a certain inductive argument, we divide $\{z_n\}$ into subsequences such that

$$\{z_n\}\cap Q=\bigcup_{j=1}^\infty\Big(M^{(j)}\cup S^{(j)}\Big),$$

where the subsequences $M^{(j)}$ and $S^{(j)}$ satisfy the following properties:

(a) Concerning $S^{(j)}$, we have

$$\sum_{z_n\in S^{(1)}}(1-|z_n|)\leq 4\,\ell(Q),$$
$$\sum_{z_n\in S^{(j)}}(1-|z_n|)\leq\frac12\sum_{z_n\in M^{(j-1)}}(1-|z_n|),\qquad j\in\mathbb{N}\setminus\{1\}.$$

(b) Concerning $M^{(j)}$, we construct sequences $\Lambda_Q^{(j)}$ such that

$$\Lambda_Q=\bigcup_{j=1}^\infty\Lambda_Q^{(j)}\subset(\Lambda\cap 4Q)$$

can be represented as a union of two separated subsequences, and

$$\sum_{z_n\in M^{(j)}}(1-|z_n|)\leq 6\sum_{\xi\in\Lambda_Q^{(j)}}(1-|\xi|),\quad j\in\mathbb{N}.$$

It is possible that some of these subsequences are empty, and in those cases the corresponding sums in (a) and (b) are zero by definition. It is clear that the properties (a) and (b) imply

$$\begin{aligned}\sum_{z_n\in Q}(1-|z_n|)&\leq\sum_{z_n\in S^{(1)}}(1-|z_n|)+\frac32\sum_{j=1}^\infty\sum_{z_n\in M^{(j)}}(1-|z_n|)\\&\leq 4\,\ell(Q)+9\sum_{\xi\in\Lambda_Q}(1-|\xi|),\end{aligned}$$

which finishes the proof, since Λ_Q can be represented as a union of two uniformly separated sequences by (b) and the assumption (ii). □

The inductive argument in the proof of Theorem 2 is based on certain auxiliary results. The most important is the following lemma, which introduces a partition of arcs playing a significant role in our construction.

Lemma 3 (See [3]). *Let $I\subset\partial\mathbb{D}$ be a closed arc for which $0<|I|<1/8$. Suppose that $0<\varepsilon<1$, and let $\{\xi_k\}_{k=1}^K\subset\mathbb{D}$ be a finite collection of points such that $\varrho(\xi_j,\xi_k)>\varepsilon$ for any $j\neq k$. Suppose that $0\leq r<1$ is sufficiently large to satisfy $\max\{|\xi_k|\}_{k=1}^K\leq r$ and $1-r\leq|I|$. Then, there exist a constant $\eta=\eta(\varepsilon)$ with $0<\eta<1$ and a partition $I=\bigcup_{n=1}^N I_n$, which divides I into $N\leq 8K+8$ closed subarcs (having pairwise disjoint interiors) such that*

- (i) *$|I_n|\geq(1-r)/64$;*
- (ii) *any hyperbolic segment γ, which joins two points in $Q(I_n)$, satisfies $\varrho(\gamma,\{\xi_k\}_{k=1}^K)>\eta$;*

for all $n=1,\dots,N$.

APPLICATION OF THEOREM 2

Let f be a non-trivial solution of the linear differential equation (3), where A is analytic in $\mathbb{D}$. Let $0<p<\infty$ be fixed, and suppose that $|A(z)|^p(1-|z|^2)^{2p-1}\,dm(z)$ is a Carleson measure. Now $\|A\|_{H_2^\infty}<\infty$ by the subharmonicity of $|A|^p$, and hence the zero-sequence $\{z_n\}$ of f is separated. Moreover, it is implicit in the proof of [6, Theorem I] that for each pair of distinct zeros z_j and z_k there exists a point $\xi_{j,k}\in\langle z_j,z_k\rangle$ at which $(1-|\xi_{j,k}|^2)^2|A(\xi_{j,k})|>1$. Define

$$\Lambda=\Big\{\xi_{j,k}\in\langle z_j,z_k\rangle:z_j,z_k\in\{z_n\},z_j\neq z_k\Big\}.$$

The property (i) in Theorem 2 is given by the construction. To see that the property (ii) holds, let $\Lambda'=\{\xi_n'\}$ be any separated subsequence of Λ with the separation constant $0<\delta<1$. Consequently, there exists a constant $\eta=\eta(\delta)$ with $0<\eta<1$ such that the Euclidean discs $D_n=D(\xi_n',\eta(1-|\xi_n'|))$ are pairwise disjoint, and $D_n\subset 2Q$ whenever $\xi_n'\in Q$. The subharmonicity of $|A|^p$ implies that there exists a constant $C=C(\delta,p)$ with $0<C<\infty$ such that

$$\begin{aligned}\sum_{\xi_n'\in Q}(1-|\xi_n'|)&\leq\sum_{\xi_n'\in Q}(1-|\xi_n'|^2)^{2p+1}|A(\xi_n')|^p\\&\leq C\sum_{\xi_n'\in Q}\int_{D_n}|A(z)|^p(1-|z|^2)^{2p-1}\,dm(z)\\&\leq C\int_{2Q}|A(z)|^p(1-|z|^2)^{2p-1}\,dm(z)\end{aligned}$$

for all Carleson squares Q. We conclude that $\{z_n\}$ is uniformly separated by Theorem 2.

We have proved the following result, which generalizes Theorem 1.

Corollary 4 (See [3]). *If the coefficient A is analytic in $\mathbb{D}$ and $|A(z)|^p(1-|z|^2)^{2p-1}\,dm(z)$ is a Carleson measure for any constant $0<p<\infty$, then the zero-sequence of each non-trivial solution of (3) is uniformly separated.*

Note that the statement converse to Corollary 4 is false, since for each $0<p<\infty$ there corresponds an analytic function $A=A(p)$ such that $|A(z)|^p(1-|z|^2)^{2p-1}\,dm(z)$ is not a Carleson measure while

$$\lim_{|z|\to 1^-}(1-|z|^2)^2|A(z)|=0,$$

and hence each non-trivial solution of (3) has at most finitely many zeros.

By the well-known connection between solutions of (3) and locally univalent meromorphic functions [6, p. 546], Corollary 4 can be stated in the following equivalent form. Recall that, if w is meromorphic and locally univalent, then its Schwarzian derivative

$$S_w=\left(\frac{w''}{w'}\right)'-\frac12\left(\frac{w''}{w'}\right)^2$$

is analytic, and the differential equation (3) with $A=S_w/2$ admits two linearly independent solutions f_1 and f_2 such that $w=f_1/f_2$. Now, the complex a-points of w (i.e., solutions $z\in\mathbb{D}$ of $w(z)=a$) are either zeros of the solution f_1-af_2 or zeros of f_2, depending whether $a\in\mathbb{C}$ or $a=\infty$, respectively.

Corollary 5 (See [3]). *If w is meromorphic and locally univalent in $\mathbb{D}$, and $|S_w(z)|^p(1-|z|^2)^{2p-1}\,dm(z)$ is a Carleson measure for any constant $0<p<\infty$, then the complex a-points of w are uniformly separated for all $a\in\mathbb{C}\cup\{\infty\}$.*

Example 1. If w is a locally univalent function in $\mathbb{D}$ such that $\log w'$ is in BMOA, then it is easy to show that its Schwarzian derivative S_w satisfies the assumption in Corollary 5. We deduce that the preimage sequence $w^{-1}(a)$ of any point $a\in\mathbb{C}$ is uniformly separated. This fact has been proved in [4, Lemma 10], and hence Corollary 5 can be understood as a generalization of this result.

References

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