UNIVERSITY OF EASTERN FINLAND

Zero separation results for solutions of $f'' + Af = 0$

JANNE GRÖHN (Joint work with M. CHUAQUI, J. HEITTOKANGAS and J. RÄTTYÄ)

Department of Physics and Mathematics, University of Eastern Finland

ABSTRACT

We discuss the oscillation of solutions of $f''+Af=0$ by focusing on four separate situations. In the complex case A is assumed to be either analytic in the unit disc $\mathbb{D}$ or entire, while in the real case A is assumed to be continuous either on $(-1,1)$ or on $(0,\infty)$. We consider the separation of zeros of non-trivial solutions in the case that A grows beyond bounds that ensure finite oscillation.

In the complex case, we show that the growth of the maximum modulus of A determines the minimal separation of zeros of all non-trivial solutions, and vice versa. This gives rise to new concepts called *zero separation exponents*. As a by-product of these findings, we rediscover the 1955-result of B. Schwarz, which asserts that $\sup_{z\in\mathbb{D}}|A(z)|(1-|z|^2)^2<\infty$ if and only if the zero-sequences of all non-trivial solutions are separated in the hyperbolic sense. The striking plane analogue reveals that the Euclidean distance between any distinct zeros of any non-trivial solution is uniformly bounded away from zero if and only if A is a constant.

In the real case, we show that the separation of zeros of non-trivial solutions is restricted according to the growth of A, but not conversely.

INTRODUCTION

The purpose of this research is to offer a unified and consistent discussion on the oscillation of solutions of the linear differential equation

$$f''+Af=0 \qquad (1)$$

in different situations. In the real case, $A=A(x)$ is assumed to be continuous either on a finite open interval $(-1,1)$ or on a half-bounded interval $(0,\infty)$. In the complex case, $A=A(z)$ is analytic either in the open unit disc $\mathbb{D}$ or in the whole complex plane $\mathbb{C}$. Under these assumptions all zeros of all non-trivial solutions of (1) are simple.

In the cases of $(-1,1)$ and $\mathbb{D}$ the distance between distinct zeros of solutions is measured by means of the hyperbolic metric. For any complex numbers $z_1,z_2\in\mathbb{D}$, the *pseudo-hyperbolic distance* $\varrho_p(z_1,z_2)$, and the *hyperbolic distance* $\varrho_h(z_1,z_2)$, between z_1 and z_2 are given by

$$\varrho_p(z_1,z_2)=\left|\frac{z_1-z_2}{1-\overline{z}_1 z_2}\right| \quad \text{and} \quad \varrho_h(z_1,z_2)=\frac{1}{2}\log\frac{1+\varrho_p(z_1,z_2)}{1-\varrho_p(z_1,z_2)}.$$

In the cases of $(0,\infty)$ and $\mathbb{C}$ the distances between distinct zeros of solutions are given in terms of the Euclidean metric.

The proofs of the main results rest upon a method of localization providing with an effective tool that takes advantage of Sturm's comparison theorem, as well as theorems of Nehari [9] and Kraus [8].

THE REAL CASE

Theorem 1 shows that the separation of zeros of solutions of (1) is connected to the growth of the coefficient A.

Theorem 1 *Let A be a continuous function in $(-1,1)$, and let $\psi:[0,1)\to(0,1)$ be a non-increasing function such that*

$$K=\sup_{0\le x<1}\frac{\psi(x)}{\psi\left(\frac{x+\psi(x)}{1+x\psi(x)}\right)}<\infty. \qquad (2)$$

If $A(x)\big(\psi(|x|)(1-x^2)\big)^2\le M<\infty$ for all $x\in(-1,1)$, then the hyperbolic distance between any distinct zeros x_1 and x_2 of any non-trivial solution of (1) satisfies

$$\varrho_h(x_1,x_2)\ge\log\frac{1+\frac{\psi(|t_h(x_1,x_2)|)}{\max\{K\sqrt{M},1\}}}{1-\frac{\psi(|t_h(x_1,x_2)|)}{\max\{K\sqrt{M},1\}}},$$

where $t_h(x_1,x_2)$ is the hyperbolic midpoint of x_1 and x_2.

Condition (2) for the non-increasing weight function ψ is not very restrictive, because it permits ψ to either decrease arbitrarily fast or arbitrarily slowly. Each of the following conditions is sufficient to ensure (2):

- $\psi:[0,1)\to(0,1)$ is differentiable, convex and $\lim_{x\to1^-}\psi'(x)=0$;
- $\psi:[0,1)\to(0,1)$ is concave and $\lim_{x\to1^-}\psi(x)=0$;
- the Lipschitz condition $\sup_{0<s<t<1}\left|\frac{\psi(s)-\psi(t)}{s-t}\right|<1$ is satisfied.

However, there are non-increasing differentiable functions ψ for which (2) fails. The functions ψ and Ψ in Theorems 2-5 have similar properties.

It is well-known that the separation of zeros of non-trivial solutions of (1) does not restrict the growth of the coefficient A. This follows from [1, Lemma 1], which implies that (1) is disconjugate whenever $\int_{-1}^{1}\max\{A(x),0\}\,dx\le2$. Therefore, if A is chosen appropriately, then $\max_{|x|\le r}A(x)$ exceeds any pregiven function in growth, while all non-trivial solutions of (1) vanish at most once.

Theorem 2, which concerns the case of $(0,\infty)$, is analogous to Theorem 1.

Theorem 2 *Let A be a continuous function on the interval $(0,\infty)$, and let $\Psi:(0,\infty)\to(0,1)$ be non-increasing on $[1,\infty)$ such that $\Psi(x)=\Psi(\frac{1}{x})$ for all $x\in(0,\infty)$, and*

$$K=\sup_{1\le x<\infty}\frac{\Psi(x)}{\Psi\left(x\frac{1+\Psi(x)}{1-\Psi(x)}\right)}<\infty. \qquad (3)$$

If $A(x)\big(\Psi(x)x\big)^2\le M<\infty$ for all $x\in(0,\infty)$, then the Euclidean distance between any distinct zeros x_1 and x_2 of any non-trivial solution of (1) satisfies

$$|x_1-x_2|\ge2\min\left\{(K\sqrt{4M})^{-1},1\right\}t_a(x_1,x_2)\Psi\big(t_g(x_1,x_2)\big),$$

where $t_a(x_1,x_2)$ and $t_g(x_1,x_2)$ are the arithmetic and the geometric mean value of x_1 and x_2, respectively.

THE UNIT DISC CASE

Considerations in $\mathbb{D}$ run parallel to the ones on $(-1,1)$. If

$$\sup_{z\in\mathbb{D}}|A(z)|(1-|z|^2)^2\le1,$$

then every non-trivial solution of (1) vanishes at most once in $\mathbb{D}$. In another form, this corresponds to the well-known univalence criterion of Z. Nehari [9, Theorem 1].

A discovery [10, Theorems 3 and 4] due to B. Schwarz states that the distance between distinct zeros of non-trivial solutions of (1) is uniformly bounded away from zero in the hyperbolic sense if and only if

$$\sup_{z\in\mathbb{D}}|A(z)|(1-|z|^2)^2<\infty.$$

The following result generalizes Schwarz's findings.

Theorem 3 *Let A be analytic in $\mathbb{D}$, $R\in[0,1)$, and let $\psi:[R,1)\to(0,1)$ be a non-increasing function such that*

$$K=\sup_{R^\star\le r<1}\frac{\psi(r)}{\psi\left(\frac{r+\psi(r)}{1+r\psi(r)}\right)}<\infty,$$

where

$$R^\star=\begin{cases}\frac{\psi(R)+R}{1+\psi(R)R}, & \text{if } 0<R<1,\\ 0, & \text{if } R=0.\end{cases}$$

(i) *If the coefficient A satisfies $|A(z)|\big(\psi(|z|)(1-|z|^2)\big)^2\le M<\infty$ for all $R\le|z|<1$, then the hyperbolic distance between any distinct zeros z_1 and z_2 of any non-trivial solution of (1), for which $|t_h(z_1,z_2)|\ge R^\star$, satisfies*

$$\varrho_h(z_1,z_2)\ge\log\frac{1+\frac{\psi(|t_h(z_1,z_2)|)}{\max\{K\sqrt{M},1\}}}{1-\frac{\psi(|t_h(z_1,z_2)|)}{\max\{K\sqrt{M},1\}}}. \qquad (4)$$

(ii) *Conversely, if (4) is satisfied for any distinct zeros z_1 and z_2 of any non-trivial solution of (1), for which $|t_h(z_1,z_2)|\ge R$, then the coefficient A satisfies*

$$|A(z)|\big(\psi(|z|)(1-|z|^2)\big)^2<3\max\{K^2,1\}\max\{K^2M,1\}$$

for $R^\star\le|z|<1$.

Let A be analytic in $\mathbb{D}$. We define the *zero separation exponent* of (1) as

$$\Lambda_{\mathrm{DE}}(A)=\inf\left\{q>0:\inf\frac{\varrho_p(z_j,z_k)}{(1-|t_h(z_j,z_k)|)^q}>0\right\}.$$

The infimum is taken over all zeros pairs of solutions, and we set $\Lambda_{\mathrm{DE}}(A)=\infty$, if the infimum is zero for all $q>0$.

The following result, which is a consequence of Theorem 3, underscores the linkage between existing growth results and the separation of zeros. Equivalence of (i) and (ii) is previously known by [3, Corollary 1.3, Theorem 1.4].

Corollary 4 *Let A be an analytic function in $\mathbb{D}$, and $\lambda\in(1,\infty)$. Then, the following assertions are equivalent:*

(i) $\sup_{z\in\mathbb{D}}|A(z)|(1-|z|^2)^{2\lambda+2}<\infty$;

(ii) *All non-trivial solutions f of (1) satisfy*

$$\sigma_M(f)=\limsup_{r\to1^-}\frac{\log^+\log^+\max_{|z|=r}|f(z)|}{-\log(1-r)}=\lambda;$$

(iii) $\Lambda_{\mathrm{DE}}(A)=\lambda$.

THE COMPLEX PLANE CASE

Theorem 5 *Let A be entire, $R\in[0,\infty)$, and let $\Psi:[R,\infty)\to(0,\infty)$ be a non-increasing function such that*

$$K=\sup_{R^\star\le r<\infty}\frac{\Psi(r)}{\Psi\big(r+\Psi(r)\big)}<\infty,$$

where

$$R^\star=\begin{cases}R+\Psi(R), & \text{if } 0<R<\infty,\\ 0, & \text{if } R=0.\end{cases}$$

(i) *If the coefficient A satisfies $|A(z)|\,\Psi(|z|)^2\le M<\infty$ for all $R\le|z|<\infty$, then the Euclidean distance between any distinct zeros z_1 and z_2 of any non-trivial solution of (1), for which the Euclidean mid-point $|t_a(z_1,z_2)|\ge R^\star$, satisfies*

$$|z_1-z_2|\ge\frac{2\Psi\big(|t_a(z_1,z_2)|\big)}{\max\{K\sqrt{M},1\}}. \qquad (5)$$

(ii) *Conversely, if (5) is satisfied for any distinct zeros z_1 and z_2 of any non-trivial solution of (1), for which $|t_a(z_1,z_2)|\ge R$, then the coefficient A satisfies*

$$|A(z)|\,\Psi(|z|)^2\le3\max\{K^2,1\}\max\{K^2M,1\},\quad |a|\ge R^\star.$$

The case $n=0$ in Corollary 6 can be considered as a plane analogue of Schwarz's classical unit disc result [10, Theorems 3 and 4].

Corollary 6 *Let A be entire. The coefficient A is a polynomial of degree n if and only if $|z_1-z_2|\big(1+|z_1+z_2|/2\big)^{n/2}$ is uniformly bounded away from zero for any distinct zeros $z_1,z_2\in\mathbb{C}$ of any non-trivial solution of (1).*

Let A be entire. We define the *zero separation exponent* of (1) as

$$\Upsilon_{\mathrm{DE}}(A)=\inf\left\{q>1:\inf|z_j-z_k|\big(1+|t_a(z_j,z_k)|\big)^{q-1}>0\right\}.$$

The infimum is taken over all zeros pairs of solutions, and we set $\Upsilon_{\mathrm{DE}}(A)=\infty$, if the infimum is zero for all $q>1$.

The following result emerges as a corollary of Theorem 5. Note in Corollary 7 that not all values $\mu\in[1,\infty)$ are permitted, since the degree of the polynomial coefficient must be an integer. It is well-known that the conditions (i)-(iii) in Corollary 7 are equivalent; see [4, Theorem 5], [6, Corollary 1.4], [7, Proposition 5.1], and [5, Corollary 3].

Corollary 7 *Let A be entire and $\mu\in[1,\infty)$. Then, the following assertions are equivalent:*

(i) *Coefficient A is a polynomial of $\deg(A)=2\mu-2$;*

(ii) *All non-trivial solutions f of (1) satisfy*

$$\rho(f)=\limsup_{r\to\infty}\frac{\log\log\max_{|z|=r}|f(z)|}{\log r}=\mu;$$

(iii) *Zeros $\{z_n\}_{n=1}^\infty$ of all non-trivial solutions f of (1) satisfy*

$$\mu(f)=\inf\left\{\beta>0:\sum_{n=1}^{\infty}|z_n|^{-\beta}<\infty\right\}\le\mu,$$

and there is a solution f of (1) such that $\mu(f)=\mu$;

(iv) $\Upsilon_{\mathrm{DE}}(A)=\mu$.

References

[1] M. Chuaqui, Integral conditions on the Schwarzian for curves to be simple or unknotted, Proc. Amer. Math. Soc. 138 (2010), no. 5, 1811–1819.

[2] M. Chuaqui, P. Duren, B. Osgood and D. Stowe, Oscillation of solutions of linear differential equations, Bull. Austral. Math. Soc. 79 (2009), 161–169.

[3] I. Chyzhykov, J. Heittokangas and J. Rättyä, Sharp logarithmic derivative estimates with applications to ODE's in the unit disc, J. Austral. Math. Soc. 88 (2010), 145–167.

[4] G. G. Gundersen, On the real zeros of solutions of $f''+A(z)f=0$ where $A(z)$ is entire, Ann. Acad. Sci. Fenn. Ser. A I Math. 11 (1986), no. 2, 275–294.

[5] G. G. Gundersen, Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates, J. London Math. Soc. 37 (1988), no. 1, 88–104.

[6] J. Heittokangas and J. Rättyä, Zero distribution of solutions of complex linear differential equations determine growth of coefficients, Math. Nachr. 284 (2011), 412–420.

[7] I. Laine, Nevanlinna Theory and Complex Differential Equations, Walter de Gruyter, Berlin, 1993.

[8] W. Kraus, Uber den zusammenhang einiger characteristiken eines einfach zusammenhängenden bereiches mit der kreisabbildung, Mitt. Math. Sem. Giessen 21 (1932), 1–28.

[9] Z. Nehari, The Schwarzian derivative and schlicht functions, Bull. Amer. Math. Soc. 55 (1949), 545–551.

[10] B. Schwarz, Complex nonoscillation theorems and criteria of univalence, Trans. Amer. Math. Soc. 80 (1955), 159–186.