Difference Painlevé Equations and Nevanlinna Theory

UNIVERSITY OF EASTERN FINLAND

Risto Korhonen

Department of Physics and Mathematics, University of Eastern Finland

Abstract

The Painlevé property is closely connected to differential equations that are integrable via related iso-monodromy problems. Many apparently integrable discrete analogues of the Painlevé equations have appeared in the literature. The existence of sufficiently many finite-order meromorphic solutions appears to be a good analogue of the Painlevé property for discrete equations, in which the independent variable is taken to be complex. If $w(z)$ is an admissible finite-order meromorphic solution of the difference equation

$$w(z+1)+w(z-1)=R(z,w(z)) \qquad (\dagger)$$

where $R(z,w(z))$ is rational in $w(z)$ with coefficients that are meromorphic in z, then either $w(z)$ satisfies a difference linear or Riccati equation or else equation (†) can be transformed to one of a list of canonical difference equations. This list consists of all known difference Painlevé equations of the form (†), together with their autonomous versions.

Painlevé equations

An ordinary differential equation is said to have the Painlevé Property when every solution is single valued, except at the fixed singularities of the coefficients. Painlevé [10, 11], Fuchs [3] and Gambier [4] looked at the class

$$w''=F(z,w,w')$$

where F is rational in w and w', rejecting those equations which did not have the Painlevé property. They singled out a list of 50 equations, six of which could not be integrated in terms of known functions, or transformed into another equation in the list. These equations are now known as the Painlevé equations, and they are as follows:

$$w''=6w^2+z \qquad (P_1)$$

$$w''=2w^3+zw+\alpha \qquad (P_2)$$

$$w''=\frac{(w')^2}{w}-\frac{1}{z}w'+\frac{1}{z}(\alpha w^2+\beta)+\gamma w^3+\frac{\delta}{w} \qquad (P_3)$$

$$w''=\frac{(w')^2}{2w}+\frac{3}{2}w^3+4zw^2+2(z^2-\alpha)w+\frac{\beta}{w} \qquad (P_4)$$

$$w''=\left\{\frac{1}{2w}+\frac{1}{w-1}\right\}(w')^2-\frac{w'}{z}+\frac{(w-1)^2}{z^2w}(\alpha w^2+\beta)+\frac{\gamma w}{z}+\frac{\delta w(w+1)}{w-1} \qquad (P_5)$$

$$w''=\frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-z}\right)(w')^2-\left(\frac{1}{z}+\frac{1}{z-1}+\frac{1}{w-z}\right)w'+\frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha+\frac{\beta z}{w^2}+\gamma\frac{z-1}{(w-1)^2}+\delta\frac{z(z-1)}{(w-z)^2}\right), \qquad (P_6)$$

where $\alpha,\beta,\gamma,\delta\in\mathbb{C}$. It was later confirmed that these equations indeed possess the Painlevé property.

Some properties of Painlevé equations:

- Can be written as the compatibility condition for a related linear problem, a so called *iso-monodromy* problem, which underlies the integrability of the equation
- Reductions of integrable partial differential equations, and the self dual Yang-Mills equations
- Appear in random matrix theory and the 2D Ising model
- Possess Bäcklund transformations, and special rational and Airy solutions

Nevanlinna order of growth

The order of growth is

$$\rho(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r},$$

where $T(r,f)=m(r,f)+N(r,f)$ is the characteristic function. Here

$$m(r,f)=\int_0^{2\pi}\log^+|f(re^{i\theta})|\frac{d\theta}{2\pi}, \qquad \log^+x=\max\{\log x,0\},$$

is the proximity function and

$$N(r,f)=\int_0^r\frac{n(t,f)-n(0,f)}{t}\,dt+n(0,f)\log r$$

is the counting function, where $n(r,f)$ is the number of poles of f in $\{|z|\le r\}$.

Discrete Painlevé Equations

There are many discrete equations that are considered to be integrable discrete analogues of the Painlevé equations. Here are only a few examples:

$$w_{n+1}+w_n+w_{n-1}=\frac{\alpha n+\beta+\gamma(-1)^n}{w_n}+\delta \qquad (dP_1)$$

$$w_{n+1}+w_{n-1}=\frac{\alpha n+\beta}{w_n}+\frac{\gamma+\delta(-1)^n}{w_n^2} \qquad (dP_1)$$

$$w_{n+1}+w_{n-1}=\frac{(\alpha n+\beta)w_n+\gamma+\delta(-1)^n}{1-w_n^2} \qquad (dP_2)$$

$$w_{n+1}w_{n-1}=\frac{\alpha w_n^2+\beta\lambda^n w_n+\gamma\lambda^{2n}}{(w_n-1)(w_n-\alpha)} \qquad (dP_3)$$

$$(w_{n+1}+w_n)(w_nw_{n-1})=\frac{(w_n^2-\alpha^2)(w_n^2-\beta^2)}{(w_n-\gamma n-\delta)^2-\zeta^2}, \qquad (dP_4)$$

where $\alpha,\beta,\gamma,\delta,\zeta\in\mathbb{C}$.

Some properties of discrete Painlevé equations:

- Appear as reductions of integrable lattice systems
- Have related linear (iso-monodromy) problems
- Possess Bäcklund transformations, as well as relations to Bäcklund transformations of differential Painlevé equations
- Special rational and discrete Riccati solutions
- Appear in the theory of orthogonal polynomials and in 2D quantum gravity

There is no known one property which could be described as the "discrete Painlevé property". However, there are a number of different methods which have been used, similarly as the Painlevé property in differential equations, to single out integrable discrete equations.

Methods to detect or construct discrete Painlevé equations:

- *Singularity confinement*, Grammaticos, Ramani and Papageorgiou [5].
- *Algebraic entropy*, Hietarinta and Viallet [9].
- *Existence of finite-order meromorphic solutions*, Ablowitz, Halburd and Herbst [1].
- *Algebraic geometric methods*, Sakai [14].
- *Distribution of orbit lengths in rational maps*, Roberts and Vivaldi [12].
- *Diophantine integrability*, Halburd [6].

Iteration of rational functions

According to Vojta's dictionary, the Diophantine integrability property corresponds exactly to the existence of finite-order meromorphic solutions. There are similar analogues to Vojta's dictionary for iteration of rational functions [2]. In the following theorem we look at algebraic entropy of a solution sequence consisting of rational functions, and use the polynomial degree growth as a criterion to single out a discrete Painlevé equation dP_I.

Theorem 1 (Halburd, RK) *Let $\{y_j\}_{j\in\mathbb{N}}$ be a sequence of non-constant rational functions of z solving*

$$y_{j+1}+y_{j-1}=\frac{a_jy_j^2+b_jy_j+c_j}{y_j^2},$$

where a_j, b_j and $c_j\not\equiv 0$ are non-zero algebraic functions of j. If the degree of $\{y_j\}_{j\in\mathbb{N}}$ grows at most polynomially as $j\to\infty$, then $a_j=0$, $b_j=Aj+B$ and $c_j=C+D(-1)^j$, where $A,B,C,D\in\mathbb{C}$.

Nevanlinna property

Ablowitz, Halburd and Herbst [1] considered discrete equations as delay equations in the complex plane which allowed them to analyze the equations with methods from complex analysis. The equations they consider to be of Painlevé type have sufficiently many meromorphic solutions of finite order. Ablowitz, Halburd and Herbst looked at, for instance, difference equations of the type

$$\overline{w}+\underline{w}=R(z,w), \qquad (1)$$

where R is rational in both of its arguments, and the z-dependence is suppressed by writing $w\equiv w(z)$, $\overline{w}\equiv w(z+1)$ and $\underline{w}\equiv w(z-1)$. They showed that if equation (1) has at least one non-rational finite-order meromorphic solution, then the degree of $R(z,w)$ in w is less or equal to two. The following theorem gives tight constraints also for the coefficients of (1).

Theorem 2 (Halburd, RK [8]) *If the equation*

$$w(z+1)+w(z-1)=R(z,w), \qquad (2)$$

where $R(z,w)$ is rational in w and meromorphic in z, has an admissible meromorphic solution of finite order, then either w satisfies a difference Riccati equation

$$\overline{w}=\frac{\overline{p}\,w+q}{w+p}, \qquad (3)$$

where $p,q\in\mathcal{S}(w)$, or equation (2) can be transformed by a linear change in w to one of the following equations:

$$\overline{w}+w+\underline{w}=\frac{\pi_1z+\pi_2}{w}+\kappa_1 \qquad (4)$$

$$\overline{w}-w+\underline{w}=\frac{\pi_1z+\pi_2}{w}+(-1)^z\kappa_1 \qquad (5)$$

$$\overline{w}+\underline{w}=\frac{\pi_1z+\pi_3}{w}+\pi_2 \qquad (6)$$

$$\overline{w}+\underline{w}=\frac{\pi_1z+\kappa_1}{w}+\frac{\pi_2}{w^2} \qquad (7)$$

$$\overline{w}+\underline{w}=\frac{(\pi_1z+\kappa_1)w+\pi_2}{(-1)^{-z}-w^2} \qquad (8)$$

$$\overline{w}+\underline{w}=\frac{(\pi_1z+\kappa_1)w+\pi_2}{1-w^2} \qquad (9)$$

$$\overline{w}w+w\underline{w}=p \qquad (10)$$

$$\overline{w}+\underline{w}=p\,w+q \qquad (11)$$

where $\pi_k,\kappa_k\in\mathcal{S}(w)$ are arbitrary finite-order periodic functions with period k.

Here $\mathcal{S}(w)$ denotes the field of small functions with respect to w. A non-rational meromorphic solution y of a difference equation is *admissible* if all coefficients of the equation are in $\mathcal{S}(y)$. For example, if a difference equation has rational coefficients then all non-rational meromorphic solutions are admissible.

- Equations (3), (4), (6), (7) and (9) correspond to known discrete equations of Painlevé type
- These equations can be explicitly solved in terms of elliptic functions if the coefficients are constants
- Shimomura [15] has proved existence of meromorphic solutions for certain classes of non-constant coefficients
- The assumption on finite order can be weakened somewhat [7]
- Classes of equations containing dP_3 and dP_5 have been similarly treated by O. Ronkainen [13]

References

[1] M. J. Ablowitz, R. G. Halburd, and B. Herbst, *On the extension of the Painlevé property to difference equations*, Nonlinearity **13** (2000), 889–905.

[2] A. E. Eremenko and M. L. Sodin, *Iterations of rational functions and the distribution of the values of Poincaré functions*, J. Soviet Math. **58** (1992), no. 6, 504–509.

[3] L. Fuchs, *Sur quelques équations différentielles linéares du second ordre*, C. R. Acad. Sci., Paris **141** (1905), 555–558.

[4] B. Gambier, *Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critiques fixes*, Acta Math. **33** (1910), 1–55.

[5] B. Grammaticos, A. Ramani, and V. Papageorgiou, *Do integrable mappings have the Painlevé property?*, Phys. Rev. Lett. **67** (1991), 1825–1828.

[6] R. G. Halburd, *Diophantine integrability*, J. Phys. A: Math. Gen. **38** (2005), L263–L269.

[7] R. G. Halburd, R. Korhonen, and K. Tohge, *Cartan's value distribution theory for Casorati determinants*, arXiv:0903.3236 (2009).

[8] R. G. Halburd and R. J. Korhonen, *Finite-order meromorphic solutions and the discrete Painlevé equations*, Proc. London Math. Soc. **94** (2007), no. 2, 443–474.

[9] J. Hietarinta and C.-M. Viallet, *Singularity confinement and chaos in discrete systems*, Phys. Rev. Lett. **81** (1998), 325–328.

[10] P. Painlevé, *Mémoire sur les équations différentielles dont l'intégrale générale est uniforme*, Bull. Soc. Math. France **28** (1900), 201–261.

[11] ______, *Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme*, Acta Math. **25** (1902), 1–85.

[12] J. A. G. Roberts and F. Vivaldi, *Arithmetical method to detect integrability in maps*, Phys. Rev. Lett. **90** (2003), art. no. 034102.

[13] O. Ronkainen, *Meromorphic solutions of difference Painlevé equations*, Ann. Acad. Sci. Fenn. Math. Diss. **155** (2010).

[14] H. Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Commun. Math. Phys. **220** (2001), 165–229.

[15] S. Shimomura, *Meromorphic solutions of difference Painlevé equations*, J. Phys. A **42** (2009), no. 31, 315213, 19.